Scaling and Creating Units

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One of the first things we learned in high school chemistry class was stoichiometry, and I was baffled over why we didn't learn it much earlier. The main idea is that *units cancel.* Here's a simple example: suppose a sandwich costs \$5. John earns \$8 per hour and works 20 hours per week. How many weeks does John need to work in order to purchase 400 sandwiches?

There are multiple ways to perform this calculation, but it all boils down to this: start with 400 sandwiches, and then multiply or divide the other numbers until you get weeks:

$$400 \text{ \sout{sandwiches}} \cdot \frac{5 \text{ \sout{dollars}}}{1 \text{ \sout{sandwich}}} \cdot \frac{1 \text{ \sout{hour}}}{8 \text{ \sout{dollars}}} \cdot \frac{1 \text{ week}}{20 \text{ \sout{hours}}} = 12.5 \text{ weeks.}$$

Notice how the units (i.e., sandwiches, dollars, hours) "cancel out" in a way that makes sense. For example, when we do $400 \cdot 5$, we're converting 400 sandwiches into the equivalent dollar amount. Then we convert dollars into hours (based on John's salary) and hours into weeks (based on John's work schedule). That's the main idea.

In this lesson, I'll illustrate some stoichiometry-related techniques by using them to solve actual problems that I (and I suspect many others) have encountered.

1. Scale down the units

Whenever I click on a YouTube video, I estimate the ratio of dislikes to likes. In particular, I check if it passes the "1%" rule, which simple asks if this ratio is less than 1%. If so, then the video's probably worth watching. (If it's over 99%, then it's probably worth watching too, to see why people dislike it so much.)

Of course, computing 1% of something is simple: just shift the decimal place two spots left. For example, suppose a video has 9300 likes. Then to pass the 1% rule, the video needs to have fewer than 93 dislikes.

However, things get a bit trickier when the numbers get bigger. This is when it becomes useful to *scale down the units.* For example, suppose a video has 4.5 million likes and 15 thousand dislikes. (These values are often displayed as "4.5M" and "15K".) Quick — what is the ratio of dislikes to likes?

Notice that a million is a "thousand thousand," so 4.5M is equal to 4.5KK. This allows us to "cancel" the "K" as if it were a unit:

$$\frac{15 \text{ K}}{4.5 \text{ M}} = \frac{15 \text{ \sout{K}}}{4.5 \text{ \sout{K}K}} = \frac{15}{45 \cdot 100} = \frac{1}{300} = \frac{1}{3} \cdot 1\%.$$

As we can see, the ratio is one third of 1%, so this video passes the 1% rule. Of course, this approach is essentially equivalent to noting that $1\text{M} = 10^6$ and $1\text{K} = 10^3$, and then canceling three zeros. But numbers in real life usually aren't presented in scientific notation, so knowing that 1M = 1KK is often useful. (Similarly, 1B = 1KM and 1T = 1MM.)

A million dollars per person?

In March 2020, a miscalculation involving a ratio went viral on national television. The network aired a tweet that claimed the following: since Michael Bloomberg spent $500 million on advertisements, and the United States population is 327 million, he could have given every American $1 million and still had leftover money.

Of course, the correct calculation cancels out the "M":

$$\frac{500 \not{\text{M}} \text{ dollars}}{327 \not{\text{M}} \text{ people}} = \frac{500 \text{ dollars}}{327 \text{ people}} \approx \frac{1.5 \text{ dollars}}{\text{person}}.$$

And what if the numerator were instead $500 billion? (For reference, in 2020, the richest person in the world has a net worth of around $180 billion.) Well, 1B = 1KM, so all we have to do is multiply the numerator by 1000. So if this money were split across all Americans, each person would get about $1500.

2. Create your own units

Another way of simplifying calculations is by changing the units themselves (rather than simply canceling a bunch of zeros from the top and bottom). For example, we usually say a building is 10 stories tall, rather than 150 feet. In other words, a "story" *is an unofficial unit* that is roughly equivalent to 15 feet.

Here's an example from cooking. A typical recipe for cooking rice requires $1\frac{1}{4}$ cups of water for every cup of rice. The calculation is trivial if we want to cook 1 cup of rice, but what if we have a quarter-cup measuring cup and want to cook 3/4 cups of rice?

Our gut reaction is to multiply 3/4 by $1\frac{1}{4}$, but this is pretty messy (even if you convert $1\frac{1}{4}$ into 5/4). The trick is to use your measuring cup as a single unit: let's define a "scoop" as 1/4 cup, so to cook 3 scoops of rice, we need $3\frac{3}{4}$ scoops of water. We don't need to convert the final answer into cups, and this trick works regardless of the size of a scoop!

$$\frac{3}{4} \text{ cup rice} \cdot \frac{1\frac{1}{4} \text{ cups water}}{1 \text{ cup rice}} = 3 \text{ \sout{scoops rice}} \cdot \frac{1\frac{1}{4} \text{ scoops water}}{1 \text{ \sout{scoop rice}}} = 3\frac{3}{4} \text{ scoops water}.$$

Finally, let's look at an example from currency exchange. The currency of South Korea is the won, and in the 2010's, the exchange rate hovered slightly above 1000 won per US dollar. This makes conversions fairly simple for somebody accustomed to USD shopping in South Korea: simply divide everything by 1000.

However, communicating these large numbers can be a bit clunky. For example, "10 to 15 dollars" becomes "10 thousand to 15 thousand won," which is quite a mouthful. To alleviate this problem, we can use the "kilowon," also known as a *kwon* (which happens to be a common Korean surname). As you might guess from the prefix, a kilowon is defined as 1000 won. This made-up unit is roughly equivalent to one US dollar, has a very natural name, and it's easy to say; the mouthful from before becomes "10 to 15 kwon." (Just to be clear, I don't think this is an actual unit of measurement. But maybe it should be.)

What's my running pace?

The speed on a treadmill is usually given as *miles per hour* (mph), but the pace of a run is often specified as *minutes per mile* (mmp[a]). So, for example, if the treadmill says "6 mph" then many runners interpret that as a pace of

$$\frac{1\ \cancel{\text{hour}}}{6\ \text{miles}} \cdot \frac{60\ \text{minutes}}{1\ \cancel{\text{hour}}} = \frac{10\ \text{minutes}}{1\ \text{mile}},$$

which is pronounced as "10 minute mile pace" (hence the abbreviation as mmp). Similarly, 10 mph is equivalent to 6 mmp. What about 8 mph?

I'll admit it: when I first encountered this situation, my gut instinct was to "interpolate" the conversion rate as if it were a line passing through $(6, 10)$ and $(10, 6)$. Under this assumption, 8 mph would be equivalent to 8 mmp.

This approach is tempting, but incorrect! In reality, x mph is equivalent to $60/x$ mmp, so the relationship between mph and mmp is not a straight line. So the correct calculation is that 8 mph is equivalent to 7.5 mmp.

And even though there isn't a straight-line relationship, there is some symmetry: x mph is equivalent to $60/x$ mmp, and y mmp is equivalent to $60/y$ mph. So from the previous calculation, we can conclude that 8 mmp is equivalent to 7.5 mph.

[a] Although the measurement is widely used, this abbreviation is not.